Quantum Chaotic Environments, the Butterfly Effect, and Decoherence

Zbyszek P. Karkuszewski, Christopher Jarzynski and Wojciech H. Zurek

Los Alamos National Laboratory, Theoretical Division, Los Alamos, New Mexico 87545, USA

We investigate the sensitivity of quantum systems that are chaotic in a classical limit, to small perturbations of their equations of motion. This sensitivity, originally studied in the context of defining quantum chaos, is relevant to decoherence in situations when the environment has a chaotic classical counterpart.

The exponential divergence of two trajectories, evolving under identical equations of motion from slightly different initial conditions – the famous butterfly effect – is a fingerprint of chaos in classical mechanics. However, an analogous definition of "quantum chaos" based on evolution in Hilbert space is problematic: by unitarity, the overlap between two evolving wavefunctions a natural indicator of distance between them – is preserved with time, hence there is no divergence. To address this difficulty, Peres [1] has suggested an alternative approach, in which one considers two trajectories evolving (in phase space or Hilbert space) from identical initial conditions but under slightly different equations of motion, rather than the other way around. Classically, even for small perturbations [2], one generically expects rapid divergence when the systems are chaotic according to the usual definition, as the perturbation (i.e. the difference between equations of motion) soon introduces a small displacement between the trajectories. Quantally, the overlap between the wavefunctions begins at unity, then decays with time, and Peres suggested the rate of this decay – a measure of the sensitivity of quantum evolution to perturbations in the equations of motion – as a signature of quantum chaos.

The sensitivity of quantum evolution also plays an important role in the context of *environment-induced decoherence* [5,6]. As an illustrative example, consider a composite system consisting of a two-state *spin* (s) and a generic *environment* (\mathcal{E}) , governed by a Hamiltonian of the form

$$\hat{\mathcal{H}} = \hat{H}_{\mathcal{E}} \otimes I_s + \hat{V}_{\mathcal{E}}^+ \otimes |+\rangle \langle +|_s + \hat{V}_{\mathcal{E}}^- \otimes |-\rangle \langle -|_s.$$
(1)

Here the identity I_s and the two projection operators act on the Hilbert space of the spin, whereas $\hat{V}_{\mathcal{E}}^{\pm}$ and $\hat{H}_{\mathcal{E}}$ act on that of the environment. We view $\hat{H}_{\mathcal{E}}$ as the "bare" Hamiltonian for the environment, and $\hat{V}_{\mathcal{E}}^{\pm}$ as a perturbative coupling to the state of the spin. An initial state $\Psi(0) = \psi_{\mathcal{E}}(0) \left[\alpha |+\rangle_s + \beta |-\rangle_s \right]$ evolves into

$$\Psi(t) = \alpha \psi_{\mathcal{E}}^+(t) |+\rangle_s + \beta \psi_{\mathcal{E}}^-(t) |-\rangle_s, \qquad (2)$$

where the unitary evolution of $\psi_{\mathcal{E}}^{\pm}(t)$ in the Hilbert space of the environment is generated by $\hat{H}_{\mathcal{E}} + \hat{V}_{\mathcal{E}}^{\pm}$. The initially pure state of the spin, $\alpha |+\rangle_s + \beta |-\rangle_s$, eventually becomes a mixture of the pointer states $\{|+\rangle_s, |-\rangle_s\}$ as a result of monitoring by the environment. The decay of $|\langle \psi_{\mathcal{E}}^- | \psi_{\mathcal{E}}^+ \rangle|^2$ is an indicator of this process: once this overlap becomes negligible, the state of the spin alone can be described in terms of classical probabilities rather than quantum amplitudes.

In view of these considerations, we are motivated to ask, what limits are placed on the sensitivity of a quantum system to perturbations in its equations of motion? The aim of this Letter is to provide answers to this question, with emphasis on systems that are chaotic in the classical limit. The object of our considerations will be a pair of wavefunctions, $\psi_1(t)$ and $\psi_2(t)$, identical at t = 0, that evolve under slightly different Hamiltonians, \hat{H}_1 and $\hat{H}_2 \equiv \hat{H}_1 + \hat{V}$, respectively. Our measure of "sensitivity" will be the rate of decay of the overlap

$$O_q(t) = |\langle \psi_1(t) | \psi_2(t) \rangle|^2.$$
 (3)

We will first derive, from the uncertainty principle, a quite general bound on the rate of this decay. We will then clarify the difference in robustness between classical chaotic systems and their quantum counterparts, in terms of the size of structures found in corresponding phase space functions (classical probability distributions and quantum Wigner functions). Finally, we will illustrate the central issues with a numerical example, placing bounds on the time needed for a (classically chaotic) quantum environment to decohere a quantum system of interest.

Quantum lower bound for overlap decay and decoherence time. Using the projection operator $\hat{P}(t) = |\psi_2\rangle\langle\psi_2|$ to rewrite the above-defined overlap as $O_q(t) = \langle\psi_1|\hat{P}|\psi_1\rangle$, and applying the Schrödinger equation in the Heisenberg picture, we obtain

$$\frac{dO_q}{dt} = -\frac{i}{\hbar} \langle [\hat{V}, \hat{P}] \rangle, \qquad (4)$$

where $\langle \cdots \rangle \equiv \langle \psi_1 | \cdots | \psi_1 \rangle$. Now, the uncertainty relation for \hat{V} and \hat{P} is $\Delta V \Delta P \geq |\langle [\hat{V}, \hat{P}] \rangle|/2$, where $(\Delta V)^2 = \langle \hat{V}^2 \rangle - \langle \hat{V} \rangle^2$ is the variance of the operator \hat{V} in the state ψ_1 , and similarly $(\Delta P)^2 = \langle \hat{P}^2 \rangle - \langle \hat{P} \rangle^2 = O_q - O_q^2$. Combining this with (4) gives

$$-\frac{dO_q}{dt} \le \left|\frac{dO_q}{dt}\right| \le \frac{2}{\hbar} \Delta V (O_q - O_q^2)^{1/2},\tag{5}$$

leading, after some algebra, to the inequality

$$O_q(t) \ge \cos^2\left(\frac{1}{\hbar}\int_0^t \Delta V \mathrm{d}t'\right) \equiv \cos^2\phi(t),$$
 (6)

valid until $\phi(t)$ (which never decreases) reaches $\pi/2$. Note that by reversing the roles of ψ_1 and ψ_2 in this argument, we typically obtain a quantitatively different, though equally valid, result. We can therefore view ΔV appearing in (6) as the spread of \hat{V} in either state ψ_1 or ψ_2 , whichever gives the tighter bound.

When ψ_1 and ψ_2 represent states of a quantum environment (as discussed above), then (6) gives the following lower bound on the decoherence times:

$$\tau_D \gtrsim \pi \hbar / 2 \overline{\Delta V},$$
 (7)

where $\overline{\Delta V}$ is to be interpreted as the typical value of ΔV during the decoherence process.

Quantum and classical overlap in terms of phase space distributions. Apart from studying the sensitivity of quantum evolution in its own right, we would like to compare it with classical sensitivity, particularly in the case of chaotic evolution. We will work with functions in phase space as these transparently suggest a classical counterpart of the quantum overlap $O_a(t)$.

Equation 3 can be rewritten as [7]

$$O_q(t) = 2\pi\hbar \int W_1(x, p, t) W_2(x, p, t) \, \mathrm{d}x \, \mathrm{d}p,$$
 (8)

where the W_i 's are Wigner functions corresponding to ψ_1 and ψ_2 , evolving under \hat{H}_1 and \hat{H}_2 . Let us now consider two classical phase space distributions, $L_1(x, p, t)$ and $L_2(x, p, t)$, obeying the Liouville equation under the respective classical Hamiltonians $H_1(x, p)$ and $H_2(x, p)$. Let us furthermore set the initial conditions for the *L*'s to be the same as those for the *W*'s: $L_1 = L_2 = W_1 = W_2$ at t = 0. In view of (8) it is now natural to define a classical overlap,

$$O_c(t) = 2\pi\hbar \int L_1(x, p, t) L_2(x, p, t) \, \mathrm{d}x \, \mathrm{d}p,$$
 (9)

where the (arbitrary) normalization factor $2\pi\hbar$ was chosen so that $O_c(0) = O_q(0) = 1$. By comparing the decay times of $O_q(t)$ and $O_c(t)$, we now have a setup for comparing quantum and classical sensitivity to perturbations in the equations of motion. Because of the relevance to decoherence, we will refer to these as the quantum and classical *decoherence times*, though in the classical case this is just convenient nomenclature.

The smallest structures of phase space distributions. A central hypothesis of this Letter is that the time scale for the decay of the overlaps O_c and O_q is determined primarily by the size of the smallest structures in the corresponding phase space distributions, with particular relevance when the classical evolution is chaotic. In both cases we have two initially identical phase space distributions (the L's or W's) evolving with time while slowly accumulating a relative displacement due to the perturbation; a substantial decay of overlap occurs when this displacement is large enough that the two functions no longer "sit one on top of the other". Clearly, this depends not only on the rate at which the functions move apart, but also on the local smallness of their structure, as this determines the degree of displacement needed to kill the overlap. The difference between the decay of overlap in the classical, chaotic case, and in its quantum counterpart, arises because of the qualitatively different mechanisms governing the emergence of small-scale details in the corresponding phase space distribution.

In the classical case, the size of local structure in L_1 and L_2 shrinks exponentially with time, due to the stretching and folding associated with chaotic evolution: the probability distributions become thin and elongated, with a local width decreasing as $\exp(-\lambda t)$, where λ is the largest Lyapunov exponent. As there is no lower bound to this smallness, it is clear that the decay time will be set predominantly by the Lyapunov time. By contrast, there are limits on the fineness of detail that can develop in the Wigner function, W; e.g. the Wigner function of a superposition of two identical Gaussians separated by ΔX – a Schrödinger cat-like state – exhibits interference fringes in momentum on a scale $\delta p \simeq \hbar/\Delta X$ [5]. More generally, when spread over an area $A = \Delta X \Delta P$ in twodimensional phase space, W exhibits local structure on scales $\delta p \simeq \hbar / \Delta X$, $\delta x \simeq \hbar / \Delta P$ [8,9]. The corresponding phase space scale is associated with the sub-Planck action $a \sim \hbar^2 / A$ which has physical consequences [9]. Most notably (in the present context), the decay of O_q occurs when the relative displacement of W_1 and W_2 is sufficient for their respective smallest-scale fringes to interfere destructively.

Two examples serve to build up intuition related to these issues.

Example I. Let us assume that identical wave functions ψ_1 and ψ_2 are superpositions of N Gaussians $\tilde{G}_j \sim \exp(-(x-x_j)^2/(2\hbar)) \exp(ip_j x\hbar)$

$$\psi_1(x) = \psi_2(x) = \sum_{j=1}^N \tilde{G}_j(x; x_j, p_j)$$

The corresponding Wigner functions W_1 and W_2 consist of N coherent-state Gaussians G_j , centered at points (x_j, p_j) , as well as pairwise interference terms $G_{j,k}$:

$$W = \sum_{j=1}^{N} G_j + \sum_{j < k} G_{j,k}.$$

We assume (following [9]) that the coherent-state Gaussians are sparse: each pair G_j and G_k is well separated by the "distance" $d_{j,k}$ phase space. The interference term $G_{j,k}$ is then another Gaussian located halfway between G_j and G_k , modulated by an oscillatory factor of frequency $d_{j,k}/\hbar$ and twice the amplitude of G_j [5]. The overlap (8) of W_1 and W_2 then works out to be $O_q \approx Ng + (N-1)Ng$, where $g = \int G_i^2 dx dp$. The first term corresponds to the overlap deposited in the coherent Gaussians G_j , the second in the interference terms $G_{j,k}$. Thus, for large N most of the overlap resides in the interference terms. If we now displace one of the WF's relative to the other, by a distance at least on the order of the size of a typical interference fringe (but small compared to the size of the G_j 's), then the contributions to the overlap from the $G_{j,k}$'s will typically interfere destructively, resulting in a total overlap $\sim 1/N \ll 1$. This is somewhat counterintuitive: for a fixed number of Gaussians of fixed size, we can increase the sensitivity of the system – as measured by the perturbation needed to kill the overlap $|\langle \psi_1 | \psi_2 \rangle|^2$ – simply by increasing the average distance between the Gaussians, or equivalently the total area occupied in phase space.

Example II. While classical probability distributions have no interference fringes, there is no bound on the smallness of structures resulting from chaotic stretching and folding. Let us examine a simplified model of the classical evolution of two Gaussian distributions in phase space, initially identical:

$$L_1(x, p, t = 0) = L_2(x, p, t = 0) = \frac{1}{2\pi\sigma_x\sigma_p} e^{-\frac{x^2}{2\sigma_x^2} - \frac{p^2}{2\sigma_p^2}},$$

with x and p dimensionless. We now assume that with time both L's are exponentially stretched in the p direction and squeezed in the x direction, in an areapreserving way: $\sigma_p(t) = \sigma \exp(\lambda t)$, $\sigma_x(t) = \sigma \exp(-\lambda t)$; furthermore, while the centroid of L_1 remains fixed at x = p = 0, the centroid of L_2 drifts with a constant velocity $\mathbf{v} = (v_x, v_p)$. These assumptions mock up the relevant features of chaotic evolution under slightly different Hamiltonians, where $|\mathbf{v}|$ is an indicator of the size of the perturbation. A simple calculation gives us the following decay of the overlap between L_1 and L_2 :

$$O_c(t) = e^{-(v_x t \cdot e^{\lambda t}/2\sigma)^2} e^{-(v_p t \cdot e^{-\lambda t}/2\sigma)^2}.$$
 (10)

Generically, the first factor will dominate, and the overlap will decay to negligible values on a time scale set by λ^{-1} .

While both of these examples are highly simplified, we believe they capture the essential physics. We now present numerical results illustrating actual evolution.

Numerical simulation. An example of a timedependent Hamiltonian that generates chaos in one dimension is [10]:

$$H = \frac{p^2}{2m} - \kappa \cos(x - l\sin(t)) + a\frac{x^2}{2}.$$
 (11)

For parameter values m = 1, $\kappa = 0.36$, l = 3.8 and a = 0.01, the stroboscopic Poincaré surface of section, Fig.1, consists of four islands of stability surrounded by a chaotic sea. We have simulated both quantum and classical evolution, starting from a Gaussian distribution

centered just outside the regular region (see Fig.1) at t = 0, and evolving under H until time t = T, at which point a perturbation is turned on and the evolution forks into two branches governed by the Hamiltonians

$$H_{\pm} = \frac{p^2}{2m} - \kappa \cos(x - l\sin(T + \tau)) + a\frac{(x \pm \epsilon)^2}{2}, \quad (12)$$

where $\epsilon = 0.5$ and $\tau \equiv t - T$. The perturbation is thus $V = H_+ - H_- = 2a\epsilon x$. The preparation time interval $(0 \le t \le T)$ allows the distributions to develop small structures in phase space. After the perturbation is turned on at $\tau = 0$, we monitor the decay of overlaps.



X FIG. 1. Poincaré surface of section. The black ellipse shows the initial distribution. The evolution is confined to the dark cloud surrounding the islands of stability.

Figure 2 shows how the decoherence time τ_D (defined here somewhat arbitrarily as the time τ at which the overlap decreases to a value 0.9 [11]) depends on preparation time T for quantum and classical evolution. For short preparation times, both are equally sensitive to the perturbation applied at $\tau = 0$, reflecting the fact that the size of the smallest structure is basically the same in both cases. However, once the distributions have spread over much of the dynamically accessible area in the phase space, which occurs at $T \approx 20$, the size of interference fringes in the Wigner functions saturates, resulting in more or less constant decoherence times even for long T. Note that the quantum lower bound (6) denoted by crosses in the Fig.2, with ΔV evaluated directly from the simulation, gives results very close to the actual decoherence times; this indicates that the quantum states used in our evolution are close to minimum uncertainty states with respect to the uncertainty principle mentioned after (4). In the absence of such structure saturation in the classical distributions L, the classical decoherence time continues to decrease with increasing preparation time, due to the presence of ever smaller structures. While the computational cost of the classical simulations became prohibitive for T > 30, the observed decrease in τ_D is consistent with a rapid approach toward zero.



FIG. 2. Time after which the overlap decreases to 0.9 versus preparation time for classical and quantum environments.

To further investigate the relevance of the smallest structure scale in the Wigner function to the decay of O_q , let us define the spread of the system in position as $\Delta X = (\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2)^{1/2}$, and in momentum $\Delta P = (\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2)^{1/2}$, with averages $\langle \cdots \rangle$ taken at $\tau = 0$. This translates to interference fringes of size $\delta x \approx \hbar/\Delta P$ in position and $\delta p \approx \hbar/\Delta X$ in momentum. Using our data, we have determined δx and δp for each of the fourteen preparation times T shown in Fig.2.



FIG. 3. Time after which overlap falls by 10% versus δp .

For the parameter values and form of the perturbation we have chosen, we have found that δp is more relevant than δx for the decay of $O_q(t)$. Therefore in Fig.3 we plot the dependence of τ_D on δp . The linear dependence observed for small values of δp (large T) is not unexpected. Recall that decoherence is achieved when the displacement of W_2 relative to W_1 becomes comparable to δp . In the regime of small δp , this occurs sufficiently rapidly that the value of δp does not change much during the process. Hence, given a constant rate of relative drift in the momentum direction (due to the form of our perturbation, $\hat{V} = 2a\epsilon \hat{x}$), we expect $\tau_D \propto \delta p$. On the other hand, when δp is initially large (small T), then the decoherence time will also be large, and δp itself will decrease during this time; hence, we expect in this regime to obtain values of τ_D that are smaller than suggested by the initial value of δp , in agreement with the three highest data points shown in Fig.3.

Conclusions. We have investigated the sensitivity of classical chaotic systems and their quantum counterparts to perturbations in their equations of motion. From general quantum considerations, we have derived a lower bound for the decay of $O_q(t)$. We have further argued that the sensitivity (in both the classical and quantum cases) is set by the size of the smallest structure of the related phase space functions. Finally, we have discussed the relevance of these results to the ability of quantum environments to rapidly decohere systems to which they are coupled.

We are grateful to Diego Dalvit and Salman Habib for comments on the manuscript and stimulating discussions. All calculations presented here were performed on the Avalon Beowulf cluster at Los Alamos National Laboratory. This research is supported by the Department of Energy, under contract W-7405-ENG-36.

- A. Peres, "Quantum Theory: Concepts and Methods", (Kluwer Academic Publishers, 1995).
- [2] By "small" we mean that the spectra of Lyapunov exponents for the two Hamiltonians are essentially the same. In particular, this implies that the perturbation neither breaks nor introduces symmetry. The hierarchy of small perturbations based on energy level spacing is discussed in [3]
- [3] Ph. Jacquod, P.G. Silvestrov, and C.W.J. Beenakker, arXive:nlin.CD/0107044.
- [4] M. Hillery, R.F. O'Connell, M.O. Scully and E.P. Wigner, Phys. Rep. 106, 121 (1984).
- [5] W. H. Zurek, Physics Today 44, 36 (1991).
- [6] D.Giulini et al, Decoherence and the appearance of a classical world in quantum theory, Springer, Berlin (1996).
- [7] For simplicity we will work in one degree of freedom, but the results extend easily to higher dimensions.
- [8] S.Habib, K. Shizume, and W.H. Zurek, Phys. Rev. Lett. 80, 4361 (1998).
- [9] W.H. Zurek, Nature 412, 712 (2001).
- [10] Z.P.Karkuszewski, J.Zakrzewski, W.H.Zurek, arXiv reference: nlin.CD/0012048.
- [11] This threshold, while admittedly somewhat close to unity, was motivated by computational necessity: with a lower threshold we would not have been able to obtain reliable classical estimates of τ_D for all of the points (up to T = 30) shown in Fig.2.